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§0 Set theory

What is it?: Study of the sets – i.e., collections of objects

Why?: All of modern math is founded on set theory.

Do we need to...?: Yes!

Russel's paradox:

$\Omega = \{$ all Sets S : $S \notin S \}$

Prop: Ω is an element of Ω.

True: Then, $\Omega \in \Omega$ then $\Omega \notin \Omega$. Contradiction

False: Then, $\Omega \notin \Omega$, So $\Omega \in \Omega$. Contradiction

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In this class: use intuitionistic set theory.

§1 Operations on subsets

Recall: For a set S, a subset T of S is a set T s.t. every element of T is an element of S. We denote this $T \subseteq S$.

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<u>Remark:</u> T is a subset of S
is equivalent

$$(x \in T) \implies (x \in S).$$

<u>e.g.]</u> · $[-1,0] \subseteq \{x \in \mathbb{R} : |x| \leq 1\}$

$\mathbb{P} = \{\text{prime numbers}\} \not\subseteq \{\text{even numbers}\}$

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Def'n: For $A, B \subseteq S$.

- their <u>union</u> is

$$A \cup B := \{x \in S : x \in A \text{ or } x \in B\}$$

- their <u>intersection</u> is

$$A \cap B := \{x \in S : x \in A \text{ and } x \in B\}$$

- their <u>difference</u> is

$$A \setminus B = \{x \in S : x \in A \text{ and } x \notin B\}$$

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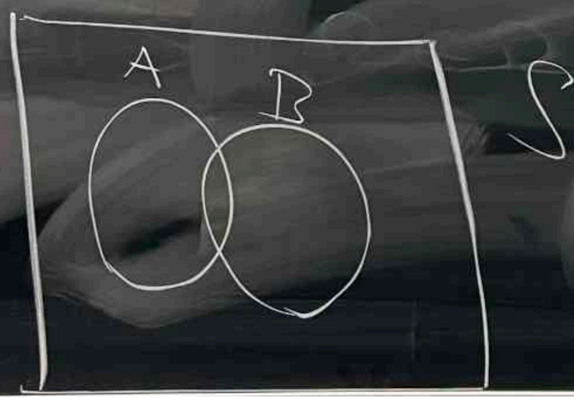
- their **symmetric difference** is

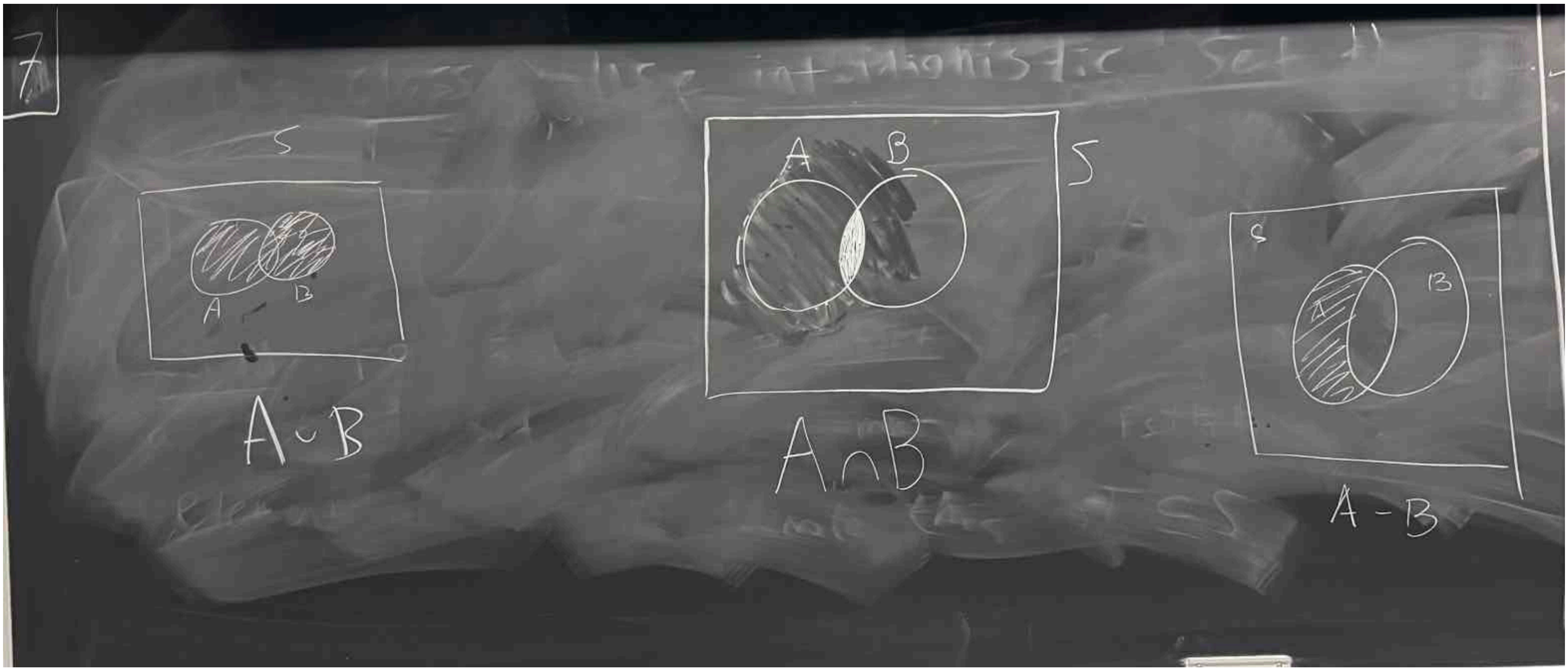
$$A \Delta B := \{x \in S : (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$$

- the **complement** of A is

$$A^c = \{x \in S : x \notin A\}.$$

Venn diagram:

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S
A
B
A ∪ B
A ∩ B
A - B

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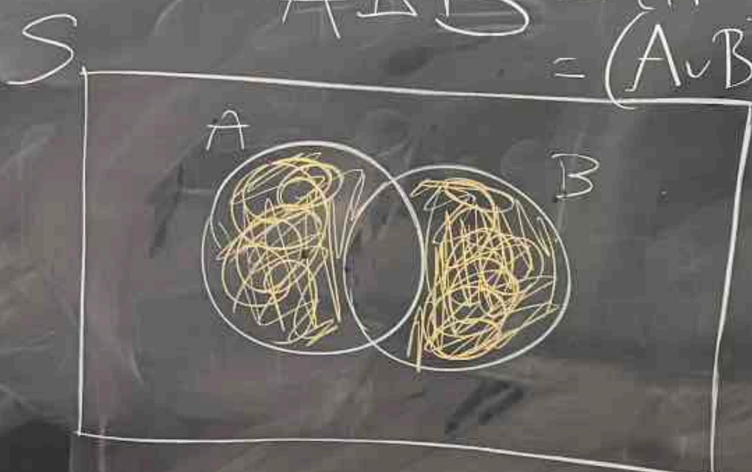
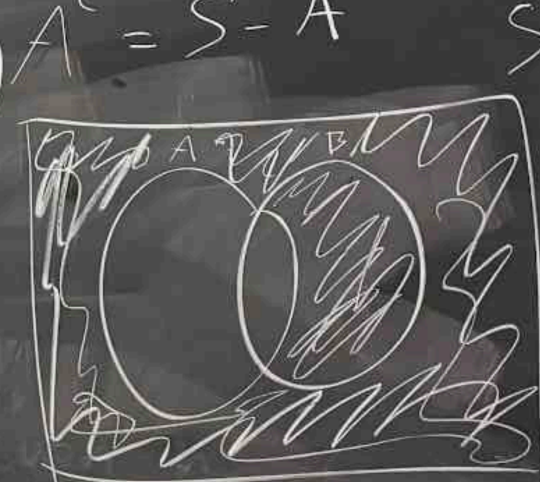
$$A \Delta B = (A-B)\cup(B-A)$$
$$= (A\cup B)-(A\cap B)$$



$$A^c = S - A$$



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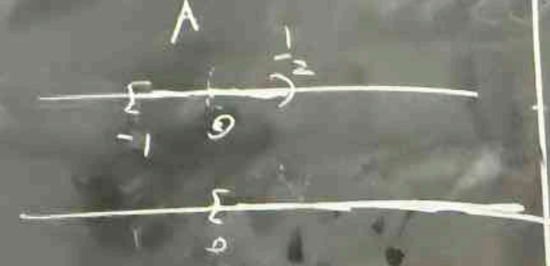
e.g.) $S=\mathbb{R}$, $A=[-1,\frac{1}{2})$, $B=\{x\in\mathbb{R}: \sqrt{x}\in\mathbb{R}\}$



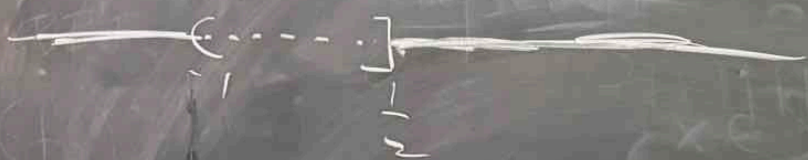
$$A\cup B=[-1,\infty)$$

$$A\cap B=[0,\tfrac{1}{2})$$

$A\triangle B$: $x\geq -1$ $x\notin[0,\frac{1}{2})$

$[-1,0)\cup[\frac{1}{2},\infty)$

$$A^c = (-\infty, -1) \cup [\tfrac{1}{2}, \infty) \qquad A \setminus B = [-1, 0)$$



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Prop: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Remark: $A = B \iff (A \subseteq B \text{ and } B \subseteq A)$

$$\iff \Big(\big((x \in A) \Rightarrow (x \in B) \big) \text{ and } \big((x \in B) \Rightarrow (x \in A) \big) \Big)$$

Mike is American
or a college student
and has a driver's license

Mike is American or is a college student
and is American or has a driver's license

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Pf: Assume $x \in A \cup (B \cap C)$. Then,

$x \in A$ or $x \in B \cap C$. So, $x \in A$ or, $x \in B$ and $x \in C$.

If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$. So,

$x \in (A \cup B) \cap (A \cup C)$. If $x \notin A$, $x \in B \cap C \subseteq (A \cup B) \cap (A \cup C)$.

Conversely, if $x \in (A \cup B) \cap (A \cup C)$, If $x \in A$, then

$x \in A \cup (B \cap C)$. If $x \notin A$, but $x \in A \cup B$ and $x \in A \cup C$,

then $x \in B$ and $x \in C$, so, $x \in B \cap C \subseteq A \cup (B \cap C)$. $\blacksquare$

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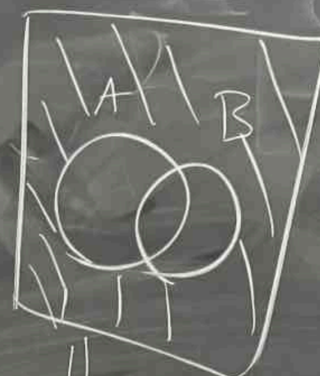
<u>Prop</u> (de Morgan's law): $(A \cup B)^c = A^c \cap B^c$.

$(A \cup B)^c$



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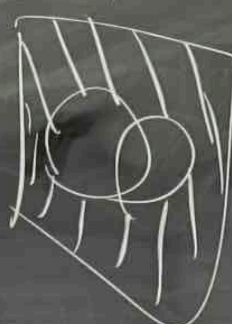
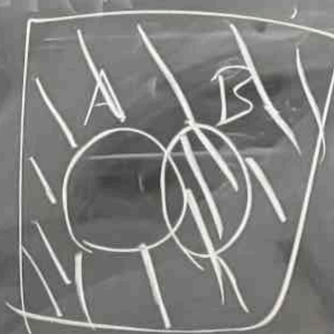
$A^c \cap B^c$



A^c



B^c

1.4

Pf: Assume $x \in (A \cup B)^c$. Then, $x \notin A \cup B$. If $x \in A$, then $x \in A \cup B$, which is false. So $x \notin A$. Similarlly, $x \notin B$. So, $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$.

Conversely, assume $x \in A^c \cap B^c$. Then, $x \in A^c$ $\boxed{\text{Xor}}$ and $x \in B^c$. So, $x \notin A$ and $x \notin B$. If $x \in A \cup B$, then $x \in A$ or $x \in B$, but this is wrong. So $x \notin A \cup B$. So $x \in (A \cup B)^c$. $\blacksquare$

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Def'n: If $\{A_i\}$ is a collection of subsets of S,

- their <u>union</u> $\bigcup_i A_i := \{x \in S : x \in A_i \text{ for some } i\}$

- their <u>intersection</u> $\bigcap_i A_i := \{x \in S : x \in A_i \text{ for all } i\}$

e.g. $\{[-x, x]\}_{\substack{x \in \mathbb{R} \\ x > 0}}$, $\bigcap_x [-x, x] = \{0\}$

$\bigcup_x [-x, x] = \mathbb{R}$

<u>Def'n</u>: For sets S and T, their <u>Cartesian product</u> is

$$S \times T := \{ (s,t) : s \in S \text{ and } t \in T \}$$

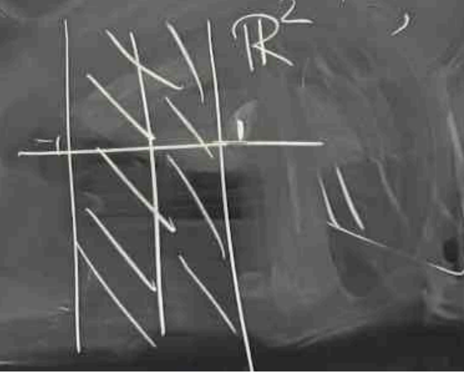
<u>e.g.</u>) $S = \{ \ddot{\smile}, \Delta \}$, $T = \{ \ddot{\frown}, O \}$, $S \times S = \{ (s,t) : s \in S, t \in S \}$

$$S \times T = \{ (\ddot{\smile}, \ddot{\frown}), (\ddot{\smile}, O), (\Delta, \ddot{\frown}), (\Delta, O) \}$$

e.g.) $S = T = \mathbb{R}$, $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$

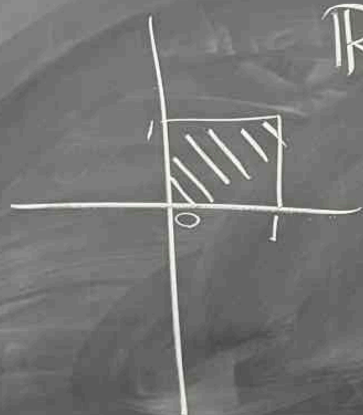
Obs: If $A \subseteq S$ and $B \subseteq T$, then $A \times B \subseteq S \times T$

e.g.) $A = [-1,1]$, $B = \mathbb{R}$ and $S = \mathbb{R}$, $T = \mathbb{R}$



e.g.) $S = T = \mathbb{R}$, $A = B = [0,1]$

$\mathbb{R}^2 = S \times T$



Exercise: $(A \cap B) \times C = (A \times C) \cap (B \times C)$.